# Game Optimal Guidance Law Synthesis for Short Range Missiles

A. Green* and J. Shinar†
*Technion—Israel Institute of Technology, Haifa 32000, Israel*
and
M. Guelman‡
*RAFAEL, Ministry of Defense, Haifa, Israel*

**A horizontal pursuit-evasion game of kind in the atmosphere between a coasting pursuer with a final velocity constraint and a maneuvering evader of constant speed is considered. For this model, which is suitable to describe short range missile engagements, the adjoint equations can be integrated analytically. This allows us to determine the optimal strategies of the players on the boundary of the capture set, called the barrier, as a function of the current and final values of the state variables. The main effect of the pursuer's final velocity constraint, an important realistic parameter, neglected in previous studies, is a substantial reduction of the capture zone. However, based on this game solution, a feedback guidance law, suitable for real-time implementation, can be synthesized and compared to other guidance laws. The results show that the corresponding capture set is much larger than the firing envelope of a similar missile guided by proportional navigation with the same final velocity constraint.**

## Introduction

IN a previous paper,[1] a coplanar (horizontal) differential game of kind between a maneuvering evader of constant speed and a coasting pursuer was analyzed. The costate equations were solved in a closed form and the optimal strategies were expressed as functions of the current and final state. In this game, no terminal constraints were imposed on the pursuer and, consequently, the game always terminated with zero closing velocity. For several reasons, such as missile maneuverability, minimum closing rate, etc., such end conditions are not acceptable in practical missile-aircraft encounters. In a subsequent paper,[2] a similar game with a constraint on the pursuer's final velocity was solved. This constraint guarantees enough maneuverability at the final phase of the interception as well as an acceptable closing velocity margin.

In both game models, the optimal strategies have an open-loop nature. They are functions of the current and the final values of the state variables. Since the final state is unknown, these strategies cannot be directly implemented. To obtain an implementable missile guidance law, the pursuer strategy has to be expressed, or at least approximated, by feedback expressions. As a first step in this direction, an attempt for guidance law synthesis based on the unconstrained solution was made.[3,4] However, such a guidance law fails at the final pursuit phase due to the zero closing velocity.

The objective of the present paper is to synthesize a feedback guidance law, based on the qualitative game solution, which is suitable for real-time implementation. For this purpose, first the solution of the coplanar differential game of kind with a realistic final velocity constraint of the pursuer[2] is briefly summarized. It is followed by a proposed feedback algorithm for guidance law synthesis. Finally, a performance comparison with an identical missile guided by proportional navigation (PN) is carried out.

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*Graduate Student, Faculty of Aerospace Engineering.
†Professor, Faculty of Aerospace Engineering. Associate Fellow AIAA.
‡Deputy Director, Guidance Division, P.O. Box 2250.

## Problem Formulation

The geometry of a planar pursuit, which defines the state variables of the game, is depicted in Fig. 1. Analysis is based on the following assumptions:

1) The pursuit takes place in the horizontal plane.
2) The pursuer is a coasting vehicle (has no thrust).
3) The evader moves with a constant speed.
4) Gravity effects on both vehicles are neglected.
5) Both vehicles have a constant minimum turning radius inversely proportional to the respective maximum lift coefficients.
6) Each player can instantaneously control his turning rate, which is proportional to the actually applied lift coefficient.
7) The pursuer has a parabolic drag polar in the well-known form

$$C_D = C_{D_o} + KC_L^2; \qquad C_L \le (C_L)_{\max} \tag{1}$$

where $C_D$ and $C_L$ are the nondimensional drag and lift coefficients, respectively. The parasitic drag coefficient $C_{D_o}$, the induced drag parameter $K$, and the maximum lift coefficient $(C_L)_{\max}$ are assumed to be constant.

Using the normalized variables

$$r \triangleq R/R_{\text{ref}} \tag{2}$$

$$v \triangleq V_P/V_E \tag{3}$$

$$t' \triangleq tV_E/R_{\text{ref}} \tag{4}$$

where $R_{\text{ref}}$ is the minimum admissible turning radius of the pursuer, the nondimensional game equations in line-of-sight coordinates are

$$\dot{r} = \cos\phi_E - v\cos\phi_P \tag{5}$$

$$\dot{\phi}_E = \sigma u_E - \dot{\theta}; \qquad |u_E| \le 1 \tag{6}$$

$$\dot{v} = -v^2(a + bu_P^2) \tag{7}$$

$$\dot{\phi}_P = vu_P - \dot{\theta}; \qquad |u_P| \le 1 \tag{8}$$

where the line-of-sight rate $\dot{\theta}$ is given by

$$\dot{\theta} = (\sin\phi_E - v\sin\phi_P)/r \tag{9}$$

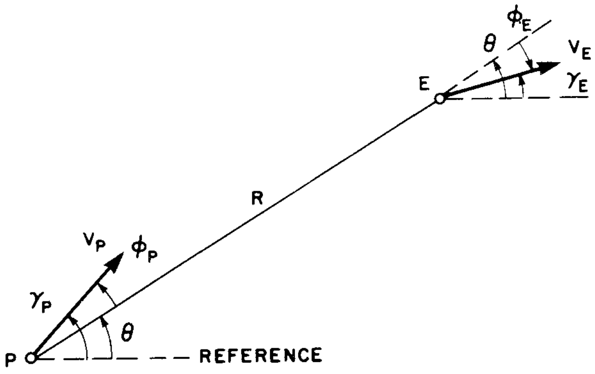


**Fig. 1 Geometry of planar pursuit.**

Moreover, $u_E$ and $u_P$ are the control variables of the evader and the pursuer, respectively, expressing for each player the ratio of the actually used lift coefficient to its maximal value; $\sigma$ is the ratio of pursuer's minimum turning radius to that of the evader; and $a$ and $b$ are the respective ratios of the parasitic and maximum induced drag coefficients to the maximum lift coefficient (both for the pursuer).

The game described by these equations terminates with capture when the pursuer approaches the evader to the normalized distance $r = r_f$ with a final velocity constraint $v \geq v_f$. No additional conditions are imposed on $\phi_E$ and $\phi_P$.

This is a qualitative pursuit evasion game. The solution of such a game is the partition of the game space into the capture zone and the escape zone, separated by a barrier hypersurface and the pair of optimal strategies of the players on this barrier.[5,6]

## Game Solution

The Hamiltonian of the qualitative game is defined as the scalar product of the time derivative of the state vector

$$X^T = [r,\ \phi_E,\ v,\ \phi_P] \tag{10}$$

and the vector $\boldsymbol{P}$

$$\boldsymbol{P}^T = [P_r,\ P_{\phi_E},\ P_v,\ P_{\phi_P}] \tag{11}$$

the latter representing the gradient of the barrier hypersurface whenever such a gradient exists. The optimal strategy pair and the respective control actions can be determined by the min-max principle of qualitative games[5,6]

$$\min_{u_P} H(X,P,u_P,u_E^*) = \max_{u_E} H(X,P,u_P^*,u_E) = H(X,P,u_P^*,u_E^*) \triangleq H^* = 0 \tag{12}$$

where an asterisk denotes the optimal control strategies. The components of the vector $\boldsymbol{P}$ have to satisfy the set of adjoint differential equations and corresponding transversality conditions

$$\dot{P}_r = -\frac{(P_{\phi_E} + P_{\phi_P})}{r^2}(\sin\phi_E - v\sin\phi_P), \qquad P_{r_f} = \mu_1 > 0 \tag{13}$$

$$\dot{P}_{\phi_E} = P_r \sin\phi_E + (P_{\phi_E} + P_{\phi_P})\cos\phi_E/r, \qquad P_{\phi_{E_f}} = 0 \tag{14}$$

$$\dot{P}_v = P_r\cos\phi_P - (P_{\phi_E} + P_{\phi_P})\sin\phi_P/r - P_{\phi_P}u_P + 2P_v v(a + bu_P^2), \qquad P_{v_f} = \mu_2 \neq 0 \tag{15}$$

$$\dot{P}_{\phi_P} = -P_r v\sin\phi_P - (P_{\phi_E} + P_{\phi_P})v\cos\phi_P/r, \quad P_{\phi_{P_f}} = 0 \tag{16}$$

From the min-max principle, the optimal control strategies on the barrier are determined

$$u_E^* = \text{sign}[P_{\phi_E}], \qquad P_{\phi_E} \neq 0 \tag{17}$$

$$u_P^* = P_{\phi_P}/(2bP_v v) \tag{18}$$

Fortunately, the adjoint equations (as in Ref. 1) can be analytically integrated in terms of the state variables and their final values (including the pursuer's final velocity). This was carried out in detail in a previous paper[2] and is summarized in Appendix A. The results for the fourth-dimensional system in line-of-sight coordinates are

$$P_r = \cos(\Delta\theta) \tag{19}$$

$$P_{\phi_E} = -\frac{1}{\sigma}[\cos\phi_{E_f} - \cos(\phi_E + \Delta\theta)]\ \text{sign}[\sin(\phi_E + \Delta\theta)] \tag{20}$$

$$P_v = -\frac{\cos\phi_{E_f}}{v}\left[\tau + \frac{1}{a}\left(\frac{1}{\bar{v}_f} - \frac{1}{v_f}\right)\right] \tag{21}$$

$$P_{\phi_P} = r\sin(\Delta\theta) - P_{\phi_E} \tag{22}$$

where $\tau$ is the normalized time to go, and

$$\Delta\theta = (\theta - \theta_f) = \int_0^\tau (\sin\phi_E - v\sin\phi_P)/r\ \mathrm{d}\tau \tag{23}$$

is the line-of-sight (LOS) angle with respect to the final LOS direction and

$$\bar{v}_f \triangleq \frac{\cos\phi_{E_f}}{\cos\phi_{P_f}} \tag{24}$$

is the pursuer's velocity on the boundary of the usable part of the target set. For a pursuer without a final velocity constraint, $v_f = \bar{v}_f$ and the game solution is identical to the one in Ref 1.

The value of $\tau$ can be explicitly expressed from the optimal Hamiltonian as a function of the state variables and their final values in the form

$$\tau = A + (A^2 + B)^{1/2} \tag{25}$$

where

$$A = \frac{1}{2a}\left[\frac{\cos(\phi_P + \Delta\theta)}{\cos\phi_{E_f}} - \frac{1}{v} + 2\left(\frac{1}{v_f} - \frac{1}{\bar{v}_f}\right)\right] \tag{26}$$

$$B = \frac{(P_{\phi_P})^2}{4ab\cos^2\phi_{E_f}} + \frac{1}{a^2}\left(\frac{1}{\bar{v}_f} - \frac{1}{v_f}\right)\left(2aA + \frac{1}{\bar{v}_f} - \frac{1}{v_f}\right) \tag{27}$$

Equation (17) indicates an eventual existence of a singular control strategy for the evader, associated with $P_{\phi_E} = 0$ for some finite period of time. Equation (14) can be written, applying Eq. (15), as

$$\dot{P}_{\phi_E} = \sin(\phi_E + \Delta\theta), \qquad P_{\phi_{E_f}} = 0 \tag{28}$$

One can thus conclude that if $\phi_{E_f} = -\theta_f$ then $\dot{P}_{\phi_{E_f}}(t_f) = P_{\phi_{E_f}} = 0$, which leads to $P_{\phi_E}(t) \equiv 0$ for at least some finite period in the neighborhood of $t_f$. The singular control strategy can be obtained by differentiating $\dot{P}_{\phi_E}$ with respect to time

$$\ddot{P}_{\phi_E} = \sigma u_E \cos(\phi_E + \Delta\theta) \tag{29}$$

and equating $\ddot{P}_{\phi_E}$ to zero along the singular trajectory. It leads to the singular evader control

$$u_E^s = 0 \tag{30}$$

implying that along the singular trajectory $\phi_E = \phi_{E_f} = -\theta_f$. This singular trajectory is on a universal surface, it reaches the target set and attracts other barrier trajectories.

Using the solution of the adjoint equations the players optimal strategies are obtained in closed form

$$u_E^* = -\operatorname{sign}[\sin(\phi_E + \Delta\theta)] \qquad (\phi_E + \Delta\theta) \neq 0$$
$$= 0 \qquad \phi_E = -\Delta\theta \tag{31}$$

$$u_P^* = \frac{P_{\phi_E} - r\sin(\Delta\theta)}{2b\cos\phi_{E_f}\{\tau + (1/a)[(1/\tilde{v}_f) - (1/v_f)]\}} \tag{32}$$

where the explicit expression of $P_{\phi_E}$ is given in Eq. (20).

The evader strategy is exactly the same as in Ref. 1, namely, a turn toward the final LOS direction. Because of the difference between $v_f$ and $\tilde{v}_f$, the pursuer's strategy is, however, slightly different. The pursuer also turns toward the final LOS direction, but the terminal constraint on the purser's velocity leads to a vanishing control at the end (since at $\tau = 0$ both $P_{\phi_E}$ and $\Delta\theta$ are zero).

A barrier construction example is carried out in the same way as before[1] by using the same parameters, which are presented in Table 1. In Fig. 2, the intersection of the barrier with the hypersurface $v = v_0$ is projected on the $\phi_{P_o} - \phi_{E_o}$ plane. It is compared to the solution of a similar game without terminal pursuer velocity constraint in Fig. 3. It can be seen that, as a result of the terminal constraint on the pursuer's velocity, the capture zone of the present game is smaller. For the numerical example used, this reduction is of the order of 10-20%. For example at a head-on initial geometry ($\phi_{P_o} = 0$ deg, $\phi_{E_o} = 180$ deg), the value of $r_o$ is reduced from 7.36 without constraint to 6.44 with $v_f = 1.5$.

## Guidance Law Synthesis

In this section, a missile guidance law, based on Eq. (32), is proposed. In this equation, the optimal strategy of the pursuer depends on four quantities ($\Delta\theta$, $\phi_{E_f}$, $v_f$, $\phi_{P_f}$), all of them unknown at any current position. For a real-time implementation of this strategy, those unknown quantities must be computed (estimated). These estimated quantities are denoted by ( ˜ ). Observing the nature of the optimal strategies, a turn toward the final LOS direction, most optimal trajectories will approach a tail chase geometry. Based on this assumption, the following approximations can be written

$$\tilde{\phi}_{E_f} \approx 0, \qquad \tilde{v}_f \approx (v)_{\min}, \qquad \tilde{\phi}_{P_f} \approx 0 \tag{33}$$

where $(v)_{\min}$ is the minimum admissible pursuer's velocity needed for adequate guidance. Following these approximations, only the value of $\Delta\tilde{\theta}$, the current LOS angle with respect to the final LOS direction, has to be computed

$$\Delta\tilde{\theta} = \theta' - \tilde{\theta}_f \tag{34}$$

where $\theta'$ and $\tilde{\theta}_f$ are the currently measured and the computed final LOS directions both with respect to some fixed frame of reference. This allows us to express the pursuer's guidance law in the form of

$$\tilde{u}_P = \tilde{u}_P(r, \phi_E, v, \phi_P, \Delta\tilde{\theta}) \tag{35}$$

The computed final LOS direction $\tilde{\theta}_f$ is a function of the initial conditions and the evader maneuver $u_E$. Since the actual maneuver of the evader cannot be predicted, a wrong value of $\tilde{\theta}_f$, based on assuming an optimal evader strategy, may lead to very poor results. Therefore, at the beginning of the engagement, the pursuer has to wait until the evader maneuver is identified and then the optimal strategy [Eq. (35)] with the correct value of $\Delta\tilde{\theta}$ can be implemented. During this waiting period, the missile cannot follow its optimal strategy and the actual capture range will be smaller than the game optimal solution. However, this value will be approached asymptotically as the waiting period approaches zero. During that period, required for identifying the evader's maneuver, some other guidance law, such as PN, can be used.

For a set of initial conditions ($r_o = 6.19$, $v_o = 2.67$, $\phi_{P_o} = 10$ deg, $\phi_{E_o} = 170$ deg), the LOS rate, assuming a pursuer guided by PN with $N = 4$, is depicted in Fig. 4. It can be seen that the LOS rate behavior is clearly different for the three main options of the evader's maneuver (maximum turn to the right,

**Table 1 Parameters of the game**

| Parameter | Symbol | Value |
|---|---|---|
| Mininum admissible turning radius of the pursuer | $R_{ref}$ | 1515 m |
| Evader velocity | $V_E$ | 300 m/s |
| Ratio of pursuer's minimum turning radius to that of the evader | $\sigma$ | 0.809 |
| Ratio of parasitic drag to maximum lift | $a$ | 0.0875 |
| Ratio of induced drag to lift at maximum lift coefficient | $b$ | 0.4 |
| Normalized final velocity of the pursuer | $v_f$ | 1.5 |
| Normalized capture range | $r_f$ | 0.033 |

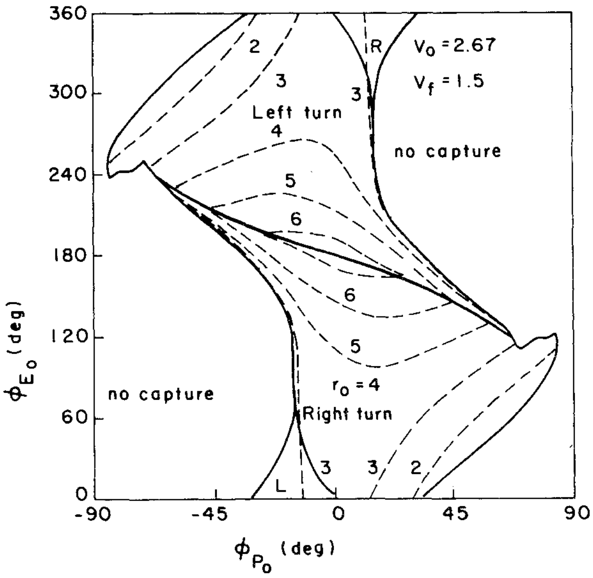


**Fig. 2 Map of maximum capture range at $v = v_o$ (view from infinite $r$).**

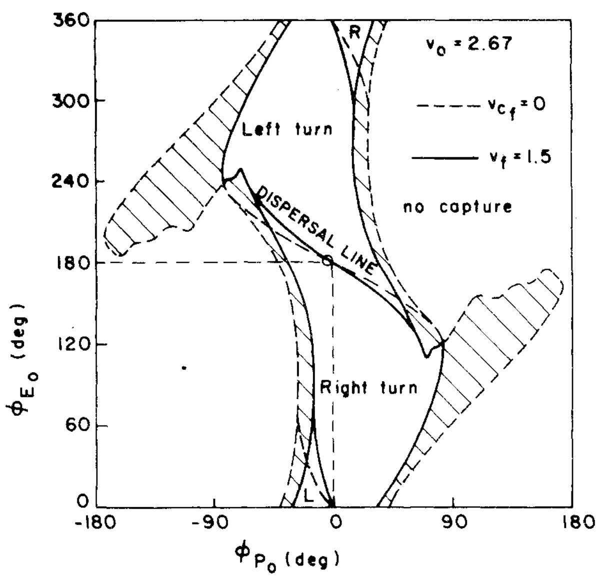


**Fig. 3 Effect of terminal pursuer velocity constraint on the capture set.**

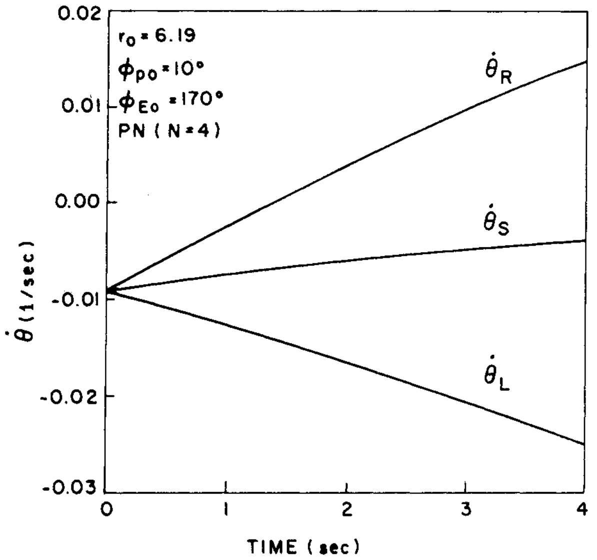


Fig. 4 Line-of-sight rate for a PN guided pursuer with $N = 4$.

maximum turn to the left, or straight line trajectory). Therefore, the LOS rate can be used as a criterion to identify the evader maneuver.

For any given evader maneuver, the value of $\tilde\theta_f$ is obtained by a fast converging iterative process. The guessed value of $\tilde\theta_f$, introduced into Eq. (35), leads to a trajectory terminating with some (generally different) value of $\theta_f$. The search algorithm has to match these two values. In this paper, a gradient method is used as a searching algorithm (see Appendix B).

Based on this discussion, the following feedback guidance law for the pursuer is proposed. This guidance law has to be robust with respect to any possible (optimal and nonoptimal) evader maneuver.

The proposed guidance law is synthesized by the following steps:

1) At $t = 0$, start guidance using PN during a fixed period $\Delta t$.

2) During that period, compute the LOS rates $\dot\theta_R$, $\dot\theta_L$, $\dot\theta_S$ (the subscripts denote right, left, and straight, respectively) obtained by the end of that period ($t = \Delta t$) by each evader maneuver option and compare it to the measured value of $\dot\theta$ at $t = \Delta t$.

3) Based on the results of the comparison take the following action:

a) If $\min\{|\dot\theta - \dot\theta_R|, |\dot\theta - \dot\theta_L|, |\dot\theta - \dot\theta_S|\} > \epsilon$, continue with PN for the next time step and go back to step 2.

b) If $|\dot\theta - \dot\theta_R|, < \epsilon$, compute $(\tilde\theta_f)_R$ (the computed final LOS direction for right turn) and $(\Delta\tilde\theta)_R$ (the corresponding LOS angle with respect to this final LOS direction) and use guidance law (31) with $\Delta\tilde\theta = (\Delta\tilde\theta)_R$.

c) If $|\dot\theta - \dot\theta_L| < \epsilon$, compute $(\tilde\theta_f)_L$ (the computed final LOS direction for left turn) and $(\Delta\tilde\theta)_L$ (the corresponding LOS angle with respect to this final LOS direction) and use guidance law (35) with $\Delta\tilde\theta = (\Delta\tilde\theta)_L$.

d) If $|\dot\theta - \dot\theta_S| < \epsilon$ compute $(\tilde\theta_f)_S$ (the computed final LOS direction for straight flying evader) and $(\Delta\tilde\theta)_S$ (the corresponding LOS angle with respect to this final LOS direction) and use guidance law (35) with $\Delta\tilde\theta = (\Delta\tilde\theta)_S$.

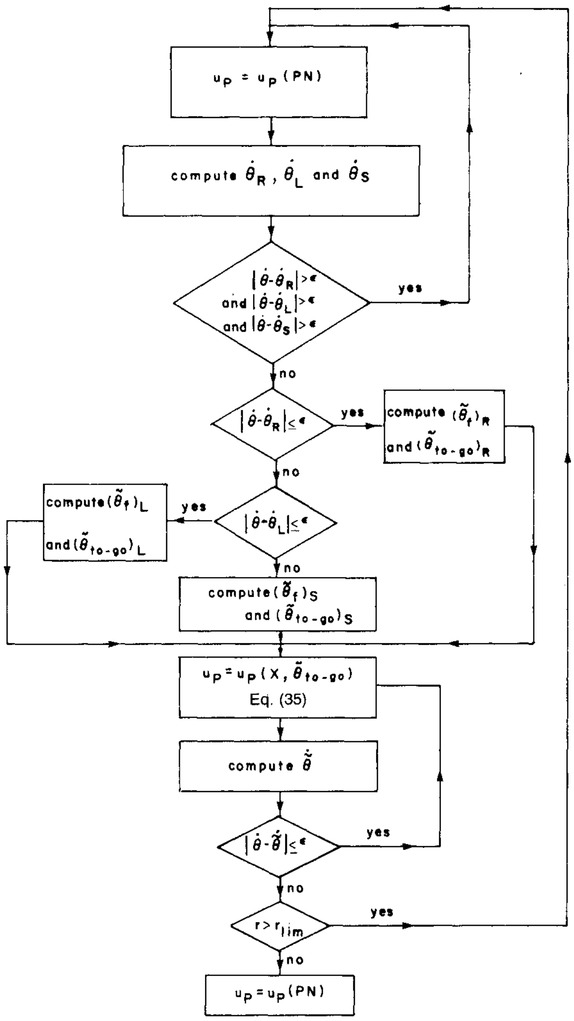


Fig. 5 Flowchart of feedback guidance law implementation.

4) Compute the expected time history of the LOS rate $\dot{\tilde\theta}$ along the assumed trajectory and compare it to the actually measured value of $\dot\theta$ at the end of each time step.

5) If $|\dot\theta - \dot{\tilde\theta}| > \epsilon$, switch to PN guidance and go back to step 1.

This guidance law closely approximates the optimal game solution for an optimal evader strategy and guarantees robustness against a nonoptimal evader behavior. However, due to the nonzero time needed to identify the actual evader maneuver and to estimate the corresponding final LOS direction, this implementation scheme may not be practical at the terminal phase of the pursuit. If in this region, defined by $r < r_{lim}$, a deviation is observed from the expected optimal game solution, a more simple feedback strategy has to be used. It can be the same PN guidance law used in the initial phase.

The proposed implementation of the feedback guidance law is summarized in the flow chart in Fig. 5. The values of $\epsilon$ and $r_{lim}$ are selected in order to guarantee a robust guidance performance. This robustness is achieved, inter alia, by respecting the terminal velocity constraints $v_f$, which guarantees a suffi-

Table 2 Comparison of maximum capture ranges

| Number | $\phi_{P_o}$, deg | $\phi_{E_o}$, deg | $t_f^*$, s | $R_o^*$, m | $\tilde R_o$, m | $\frac{\tilde R_o - R_o^*}{R_o^*}$, % | $R_o$(PN), m | $\frac{\tilde R_o - R_o(\text{PN})}{R_o(\text{PN})}$, % |
|---|---|---|---|---|---|---|---|---|
| 1 | 10 | 170 | 16.64 | 9380 | 9377 | − 0.032 | 8650 | 8.40 |
| 2 | 10 | 90 | 16.80 | 5784 | 5783 | − 0.017 | 4792 | 20.68 |
| 3 | 10 | 270 | 14.90 | 5043 | 5042 | − 0.020 | 2842 | 77.41 |
| 4 | 60 | 90 | 12.72 | 3929 | 3928 | − 0.025 | 3539 | 10.99 |

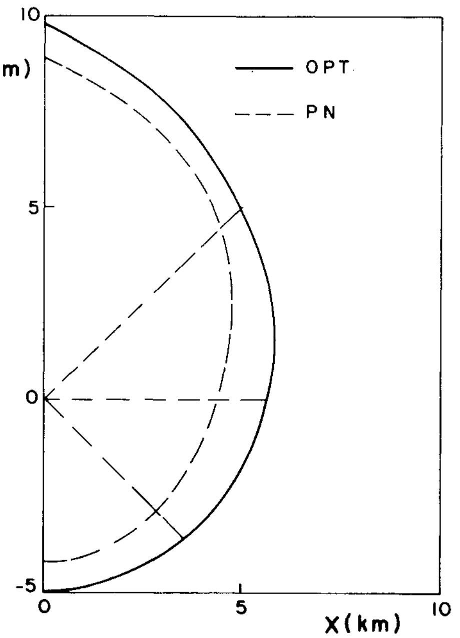


Fig. 6 Capture zone comparison (proposed guidance law vs PN).

Table 3 Comparison of maximum capture ranges ($\Delta t \neq 0$)

| $\Delta t$, s | $\tilde{R}_o$, m | $\frac{\tilde{R}_o - R_o^*}{R_o^*}$, % | $\frac{\tilde{R}_o - R_o(\text{PN})}{R_o(\text{PN})}$, % |
|---|---|---|---|
| 0.0 | 9377 | − 0.032 | 8.40 |
| 0.5 | 9328 | − 0.554 | 7.84 |
| 1.0 | 9283 | − 1.034 | 7.32 |
| 1.5 | 9243 | − 1.461 | 6.86 |
| 2.0 | 9208 | − 1.834 | 6.45 |

cient closing velocity margin needed for effective terminal homing.

Obviously, the maximum range of the proposed (suboptimal) guidance law $\tilde{R}_o$ is expected to be shorter than the capture range of the game solution.

## Numerical Results

For four examples of different initial conditions $\tilde{R}_o$, the maximum range achieved by the proposed guidance law (with $r_{\text{lim}} = 0.66$, $\epsilon = 2.5 \times 10^{-3}$, and $\Delta t \to 0$) was compared both with $R_o^*$, the maximum range of the game solution, and $R_o(\text{PN})$, the maximum range obtained by a missile guided by PN with the same terminal velocity constraint.[7] For the comparison with the PN missile, a navigation ratio of $N = 4$ was chosen because it is the lowest one that provides an efficient guidance against maneuvering targets in a scenario of nonlinear geometry.[8] The results summarized in Table 2 show that the performance loss of the proposed guidance law due to the approximation of the final quantities ($\theta_f$, $\phi_{E_f}$, $v_f$, $\phi_{P_f}$) is negligible while the improvement achieved with respect to PN is substantial. This improvement is demonstrated in Fig. 6, which compares the capture zones of the game and a PN missile with $N = 4$ for initial conditions favorable for the missile ($\phi_{P_o} = 0$).

The effect of the time delay $\Delta t \neq 0$ is presented in Table 3 for the initial conditions of example 1. These results show that the pursuer can wait as long as 1-2 s to identify the evader maneuver for computing $\Delta\tilde{\theta}$ without a substantial loss of capture range.

A large set of additional test examples[9] demonstrated that the proposed guidance law implementation is robust against all possible evader maneuvers.

## Conclusions

In this paper, a qualitative planar pursuit-evasion game between a coasting missile and a constant speed maneuvering evader, including a final velocity constraint imposed on the pursuer, is presented. Such a constraint is a necessary element in any realistic missile model. This game (similar to Ref. 1) yields a closed-form solution for the optimal strategies. These strategies are expressed as functions of the current and terminal state and allow a direct construction of the game barrier. The capture zone of the present game is obviously smaller than the one obtained without the terminal constraint on the pursuer's velocity.

Based on the new game solution, a missile guidance law implementation is synthesized. This guidance law provides a substantial performance improvement by enlarging the capture zone in comparison with an identical missile guided by PN. It was demonstrated by a large number of test examples that this implementation scheme is robust with respect to nonoptimal evader behavior and its capture range is only slightly inferior to the optimal game solution, the minor difference being proportional to the time required for identifying the actual evader maneuvers. As a consequence, the proposed guidance law can be an attractive candidate for highly maneuverable short range homing missiles, where effects of gravity, altitude variations, etc., can be neglected.

## Appendix A: Solution of Costate Equations

This solution is based on the assumption that the costate vector, associated with the gradient of the barrier surface, is continuous. Combining Eqs. (14) and (15) and using Eqs. (5) and (9) one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}(P_{\phi_E} + P_{\phi_P}) = P_r\dot{\theta} + P_{\phi_E} + P_{\phi_P}\dot{r}/r; \qquad P_{\phi_E} + P_{\phi_{P_f}} = 0 \tag{A1}$$

$$\dot{P}_r = -(P_{\phi_E} + P_{\phi_P})\dot{\theta}/r; \qquad Pr_f = \mu_1 > 0 \tag{A2}$$

These two equations can be directly integrated to yield

$$Pr = \mu_1 \cos(\Delta\theta) \tag{A3}$$

$$P_{\phi_E} + P_{\phi_P} = \mu_1 r \sin(\Delta\theta) \tag{A4}$$

where

$$\Delta\theta = -\int_0^\tau (\sin\phi_E - v\sin\phi_P)/r \,\mathrm{d}\tau \tag{A5}$$

is the LOS direction with respect to the final LOS direction and $\tau$ is the normalized time to go.

The constant of integration $\mu_1$ can be taken as 1 without losing generality. Substituting Eqs. (A3) and (A4) into Eq. (14) and rearranging, one gets

$$\dot{P}_{\phi_E} = \sin(\phi_E + \Delta\theta) \tag{A6}$$

Integration of Eq. (A6) with $|u_E| = 1$ (no singular control), taking into account Eq. (6), yields

$$|P_{\phi_E}| = (1/\sigma)\,[\cos\phi_{E_f} - \cos(\phi_E + \Delta\theta)] \tag{A7}$$

From Eq. (A4) one obtains

$$P_{\phi_P} = r\sin(\Delta\theta) - P_{\phi_E} \tag{A8}$$

Substituting Eqs. (A3), (A4), (A7), (17), and (18) into the Hamiltonian and rearranging leads to

$$H^* = \cos\phi_{E_f} - v\cos(\phi_P + \Delta\theta) - aP_v v^2 + \frac{P_{\phi_P}^2}{4bP_v} = 0 \tag{A9}$$

Multiplying Eq. (15) by $v$, and Eq. (7) by $Pv$, and adding both to $H^*$, which is zero, yields

$$(v\dot{P}_v) = \cos\phi_{E_f} \tag{A10}$$

which can be directly integrated (backward) to

$$vP_v = -\cos\phi_{E_f}\tau + \mu_2 v_f \tag{A11}$$

The constant of integration $\mu_2$ is found from $H^* = 0$ evaluated at $\tau = 0$

$$\mu_2 = \frac{\cos\phi_{E_f}}{av_f}\left(\frac{1}{v_f} - \frac{\cos\phi_{P_f}}{\cos\phi_{E_f}}\right) \tag{A12}$$

In a game without a final velocity constraint, $v_f = \cos\phi_{E_f}/\cos\phi_{P_f} \triangleq \bar{v}_f$ and, consequently, $\mu_2 = 0$. In the present case, $v_f > \bar{v}_f$ and, therefore, $\mu_2 < 0$.

This completes the integration of the costate equations as summarized in Eqs. (19–22).

## Appendix B: Final Line-of-Sight Direction Searching Algorithm

To apply the guidance law described in Eq. (35), it is necessary to compute $\Delta\tilde{\theta}$ the LOS angle with respect to the final LOS direction. As a first step, the final LOS direction must be computed. This purpose can be achieved by a search for the final LOS direction, which brings the final range between the pursuer and the evader to a minimum. For a given target maneuver $u_E$ and the approximations given in Eqs. (33), this is an optimal control problem with the following formulations:

$$Y^T = (r, \phi_E, v, \phi_P) = X^T \tag{B1}$$

$$\dot{Y} = f(Y, \theta_f) \tag{B2}$$

The vector equation (B2) is obtained by substituting $u_P$ from Eq. (35) and the selected evader strategy $u_E$ into Eqs. (5–8).

This optimization problem has two alternative formulations with different terminal constraints and performance indices (both with unspecified terminal time).

1) The performance index to be minimized is the final range

$$J_1 = r(t_f) \tag{B3}$$

with a terminal constraint on the pursuer's velocity

$$v(t_f) \geq v_f \tag{B4}$$

This formulation is suitable to initial ranges close to the capture zone boundary and the minimized final range is slightly smaller or equal to the target set radius.

2) The performance index to be maximized is the final pursuer's velocity

$$J_2 = v(t_f) \tag{B5}$$

with a terminal constraint on the final range

$$r(t_f) \leq r_f \tag{B6}$$

This alternative applied to initial ranges inside the capture zone and the maximized final pursuer's velocity is higher than the pursuer's velocity constraint.

In both cases, the required $\tilde{\theta}_f$ is found by the gradient method, based on the sensitivity functions of $\theta_f$ with respect to $r(t_f)$ for case 1 or $v(t_f)$ for case 2. First, the minimal final range is found by using formulation 1 and the following tests:

1) If the minimized value of $r(t_f)$ is bigger than the target set radius $r_f$, the missile should not be fired.

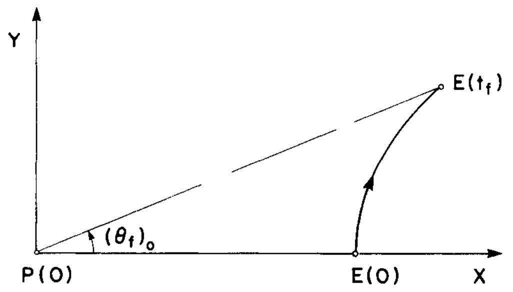


**Fig. B1 Definition of $(\tilde{\theta}_f)_o$.**

2) If the minimized value of $r(t_f)$ is equal or slightly smaller than the target set radius, the guidance law of Eq. (35) is used with the appropriate value of $\tilde{\theta}_f$.

3) If the minimized value of $r(t_f)$ is inside the target set and the pursuer's final velocity is bigger than the pursuer's minimal velocity, solve the problem with formulation 2 and use the guidance law of Eq. (35) with the appropriate $\tilde{\theta}_f$.

The iterative mechanisms to compute the final LOS direction are ($i$ denotes the iteration index)

For case 1:

$$(\tilde{\theta}_f)_{i+1} = (\tilde{\theta}_f)_i + \frac{1}{G}\left\{\frac{(\tilde{\theta}_f)_i - (\tilde{\theta}_f)_{i-1}}{[r(t_f)]_i - [r(t_f)]_{i-1}}\right\}[-r(t_f)_i] \tag{B7}$$

For case 2:

$$(\tilde{\theta}_f)_{i+1} = (\tilde{\theta}_f)_i + \frac{1}{G}\left\{\frac{(\tilde{\theta}_f)_i - (\tilde{\theta}_f)_{i-1}}{[v(t_f)]_i - [v(t_f)]_{i-1}}\right\}[\Delta v(t_f)] \tag{B8}$$

where $G$ is a gain, a function of $i$ and $\Delta v(t_f)$ is a chosen value that guarantees fine convergence in case 2. If the just mentioned sensitivity function is small in absolute value, the value of $G$ has to be large. This situation occurs when the guessed $\tilde{\theta}_f$ is far from the solution (typical to the initial stage of the iterative process). Numerical experience has shown that if the value of $G$ is too small at the beginning (e.g., $G = 1$), the iterative process diverges, whereas a larger $G$ at the beginning provides a slow, but sure convergence.

To initiate the iterative process, a PN guided missile is assumed and the equations of motion are simulated until the relevant constraint is violated. For the zero-order guess of $\tilde{\theta}_f$, the following expression is used:

$$(\tilde{\theta}_f)_o = tg^{-1}\left\{\frac{Y_E(t_f) - Y_P(0)}{X_E(t_f) - X_P(0)}\right\} \tag{B9}$$

It is based on the assumption that the value of $\tilde{\theta}_f$ is not very far away from the direction of a straight line trajectory between the pursuer's initial position and the evader's final position with respect to a fixed reference line (see Fig. B1).

Subsequently, a second value of $\tilde{\theta}_f$, rather close to $(\tilde{\theta}_f)_o$, is chosen for another simulation using the guidance law of Eq. (35). Based on those two simulations, the iterative algorithm in Eqs. (B7) and (B8) can be initiated.

The algorithm converges to a close neighborhood of the true value of $\theta_f$ ($\pm$ 1 deg) in a few steps; much faster than the time needed to identify the actual evader maneuver. During the pursuit, further refinement can be made without introducing any delay.

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